The entanglement criterion of multiqubits

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Abstract

We present an entanglement criterion for multiqubits by using the quantum correlation tensors which rely on the expectation values of the Pauli operators for a multiqubit state. Our criterion explains not only the total entanglement of the system but also the partial entanglement in subsystems. It shows that we have to consider the subsystem entanglements in order to obtain the full description for multiqubit entanglements. Furthermore, we offer an extension of the entanglement to multiqudits.

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Entanglement has been an important key word in the quantum computer and the quantum information technology. In particular, entanglements in the bipartite qubits have many applications as the superdense coding[1], quantum computation, teleportation[2], clock synchronization[3],[4]and quantum cryptography[5]. These have been clarified by a negative partial transposition [6],[7], and quantified by concurrence[8], negativity[9], entanglement of formation[10], etc..

As many entangled states such as GHZ state, W state, etc., were found in multiqubits, the multiqubit entanglements have been applied to a real physical system such as quantum secret sharing and the one-way quantum computer[11]. The investigation on the entanglement properties in the multipartite system has thus emerged as a central problem in quantum information study. However, no efficient method to clarify the status of multipartite entanglements has been introduced.

The classification of the mathematical and physical structures in multipartite states was, at first, tried by the local operation associated with classical communication(LOCC). In the multipartite systems, the investigations on entanglement measure have been proposed by the *tangle*[12] which is computed by the concurrence between two intentionally divided subsystems in an effective two-dimensional Hilbert space. Recently, entanglement witnesses were suggested by another method for the classification of multipartite entanglements[7],[13]. This method requires the witness operators to detect various forms of multipartite entanglements. However, it is difficult to know the witness operators before we classify the multipartite system, and additionally witness operators are defined by some a priori knowledge about the states under investigation.

In spite of trials, the entanglements in a multipartite system are complicated even in pure systems because the quantum states can share entanglements differently among possible subsystems and have the different classes of total entangled states as GHZ, W or cluster states. It is important to define an entanglement criterion that could distinguish all possible types of entanglements which exist among the constituents. In this letter, we present a general entanglement criterion that can solve the above problems for pure multiqubit systems. Furthermore we will discuss that our entanglement description can be extended to multiqubit mixed states and higher dimensional Hilbert spaces.

In general, a pure composite system with N qubits can be represented

by

$$|\Psi(1,2,3,\cdots,N)\rangle = \sum_{IJ\cdots K=0}^{1} a_{IJ\cdots K}|I\rangle_1 \otimes |J\rangle_2 \otimes \cdots \otimes |K\rangle_N, \qquad (1)$$

where $\sum_{IJ\cdots K=0}^{1} |a_{IJ\cdots K}|^2 = 1$ and the state is an element of composite Hilbert space as $\mathcal{H}_N = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2$.

Our question is whether this state is separable or entangled. In order to answer this question, we introduce the quantum correlation tensor[14] for the given multipartite qubit system $|\Psi\rangle$ as

$$\begin{aligned} &M_{i_1 i_2 \cdots i_n}(\alpha_1, \alpha_2, \cdots, \alpha_N; |\Psi\rangle) = \langle\Psi|(\sigma_{i_1}(\alpha_1) - \lambda_i(\alpha_1)) \otimes (\sigma_{i_2}(\alpha_2) - \lambda_{i_2}(\alpha_2)) \\ &\otimes \cdots \otimes (\sigma_{i_n}(\alpha_n) - \lambda_{i_n}(\alpha_n)) \otimes I(\alpha_{n+1}) \cdots \otimes I(\alpha_N)|\Psi\rangle, \qquad (2) \end{aligned}$$

where $n \leq N$, $\sigma_i(\alpha)$ denotes the i^{th}-component Pauli's operator of the α^{th} qubit and $\lambda_{i_j}(\alpha_j) = \langle\Psi|I(1) \otimes I(2) \otimes \cdots \sigma_{i_j}(\alpha_j) \cdots \otimes I(N)|\Psi\rangle$. Here, $I(\alpha)$ is the identity operator on the α^{th} qubit. Obviously, M determines whether one qubit is separated. If the state is separable such as $|\Psi(1,2,3,\cdots,N)\rangle = |\Psi(\alpha_1, \alpha_2, \cdots, \alpha_{N-1})\rangle \otimes |\Psi(\alpha_N)\rangle$, $M_{i_1 i_2 \cdots i_N}(1,2,\cdots,N; |\Psi\rangle)$ must be zero. This fact can be proved by a simple calculation. Conversely, if $M_{i_1 i_2 \cdots i_N}(1,2,\cdots,N;|\Psi\rangle) = 0$ for all $i_1, i_2, \cdots, i_N$, the state is separable such as $|\Psi(1,2,3,\cdots,N)\rangle = |\Psi(\alpha_1, \alpha_2, \cdots, \alpha_{N-1})\rangle \otimes |\Psi(\alpha_N)\rangle$, which implies that one of the qubits is uncorrelated with the others. We can consider a N-qubit system as a bipartite system consisted of a $(N-1)$-qubit system and a single qubit system. So the original state can be rewritten by the Schmidt's decomposition as

$$|\Psi(1,2,3,\cdots,N)\rangle = \alpha|a,0\rangle + \beta|b,1\rangle, \qquad (3)$$

where $|a\rangle$ and $|b\rangle$ are orthogonal states of the $(N-1)$-qubit system with $\alpha^2 + \beta^2 = 1$. We denote the operators of the $(N-1)$-qubit space as a generator, $\hat{L_K}$. We can calculate M for a bipartite system as following;

$$\begin{aligned} M_{Kx} &= (\alpha\langle a,0| + \beta\langle b,1|)\hat{L_K} \otimes \sigma_x(\alpha|a,0\rangle + \beta|b,1\rangle) - L_K\lambda_x \\ &= 2\alpha\beta Re(\langle a|\hat{L_K}|b\rangle), \\ M_{Ky} &= (\alpha\langle a,0| + \beta\langle b,1|)\hat{L_K} \otimes \sigma_y(\alpha|a,0\rangle + \beta|b,1\rangle) - L_K\lambda_y \\ &= 2\alpha\beta Im(\langle a|\hat{L_K}|b\rangle), \\ M_{Kz} &= (\alpha\langle a,0| + \beta\langle b,1|)\hat{L_K} \otimes \sigma_z(\alpha|a,0\rangle + \beta|b,1\rangle) - L_K\lambda_z \\ &= \alpha^2\langle a|\hat{L_K}|a\rangle - \beta^2\langle b|\hat{L_K}|b\rangle - (\alpha^2 - \beta^2)(\alpha^2\langle a|\hat{L_K}|a\rangle + \beta^2\langle b|\hat{L_K}|b\rangle), \end{aligned}$$

where $L_K = (\alpha\langle a,0| + \beta\langle b,1|)(\hat{L_K} \otimes I)(\alpha|a,0\rangle + \beta|b,1\rangle)$. α or β has to be vanished in order for all of M's to be zero. This indicates that at least one of the qubits is uncorrelated with the others.

For a two-qubit system M_{ij} is the criterion to judge whether the bipartite pure state is separated or entangled. Schlinz and Mahler suggested this scenario in a bipartite system[14]. The three-qubit state, $|\Psi(1,2,3)\rangle$, has three types of entanglements; a separated state as $A-B-C$, a bipartite entangled state as $A-BC, AB-C$ or $C-AB$ and a totally entangled state as ABC. Nonzero of any $M_{ijk}(1,2,3;|\Psi\rangle)$ tells us that the state is totally entangled as the GHZ or W state. Zero of $M_{ijk}(1,2,3;|\Psi\rangle)$ for any i,j,k means that the state can be either $|\Psi(1,2)\rangle \otimes |\Psi(3)\rangle$ or $|\Psi(1)\rangle \otimes |\Psi(2)\rangle \otimes |\Psi(3)\rangle$. However, the increase of qubit numbers in the system produces the increase of the possibilities in the entanglement types. We have to differentiate all these situations in order to fully describe the entanglement structure. When $M_{i_1 i_2 i_3 \cdots i_N}(\alpha_1, \alpha_2, \cdots, \alpha_N; |\Psi\rangle) = 0$ for all $i_1, i_2, \cdots, i_N$, there are various situations including completely separable and multi-separable cases. We can easily investigate that the state is at least one-qubit separated such as $|\Psi(\alpha_1, \alpha_2, \cdots, \alpha_{N-1})\rangle \otimes |\Psi(\alpha_N)\rangle$, but we cannot judge directly whether the state, $|\Psi(\alpha_1, \alpha_2, \cdots, \alpha_{N-1})\rangle$ is entangled or separated. $M_{i_1 i_2 \cdots i_n}(\alpha_1, \alpha_2, \cdots, \alpha_n; |\Psi\rangle)$ with $n < N$ needs to determine the entanglement of the subsystems consisted of n qubits. $M_{i_1 i_2 \cdots i_n}(\alpha_1, \alpha_2, \cdots, \alpha_n; |\Psi\rangle)$ is not zero for the states which have the entanglement among the n qubits. $M_{i_1 i_2 \cdots i_n}(\alpha_1, \alpha_2, \cdots, \alpha_n; |\Psi\rangle)$ can distinguish totally entangled state from partially entangled state such as $|\Psi(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_n})\rangle \otimes |\Psi(\alpha_{i+1})\rangle \otimes |\Psi(\alpha_{i+2})\rangle \otimes \cdots \otimes |\Psi(\alpha_N)\rangle$.

The correlation tensor, M, is sufficient in bipartite and tripartite systems. However, the situation is different in systems consisted of more than three qubits. They cannot assure a criterion for partially entangled cases such as $|\Psi(\alpha_1, \alpha_2, \cdots, \alpha_l)\rangle \otimes |\Psi(\beta_1, \beta_2 \cdots \beta_m)\rangle \otimes \cdots \otimes |\Psi(\gamma_1, \gamma_2 \cdots \gamma_n)$. We have to modify the quantum correlation tensors to solve the problem by

$$\begin{aligned}
&M'_{i_1 i_2 \cdots i_n}(\alpha_1, \alpha_2, \cdots, \alpha_n; |\Psi\rangle) = M_{i_1 i_2 \cdots i_n}(\alpha_1, \alpha_2, \cdots, \alpha_n; |\Psi\rangle) \\
&- \sum_{A \cup B \cup \cdots \cup C = \{1,2,3 \cdots n\}} (M_A(\alpha'_1, \alpha'_2, \cdots, \alpha'_{n_1}; |\Psi\rangle) M_B(\alpha''_1, \alpha''_2, \cdots, \alpha''_{n_2}; |\Psi\rangle) \\
&\cdots M_C(\alpha'''_1, \alpha'''_2, \cdots, \alpha'''_{n_m}; |\Psi\rangle), \qquad (4)
\end{aligned}$$

where $n_1 + n_2 + \cdots + n_m = n$ and $A = i_1 i_2 \cdots i_{n_1}$ and i_j denotes the i^{th}-component of Pauli' operator acted on the j^{th} qubit. The sum of the second term in the right side of eq. (4) denotes the possible disjoint partitions of indices composed of Pauli's components of each qubit. Since M' equals to M in bipartite and tripartite systems, it is requested in the systems which consist of more than three qubits. For instance, M' of the four-qubit case is

written by

$$M'_{i_1 i_2 i_3 i_4}(1,2,3,4;|\Psi\rangle) = M_{i_1 i_2 i_3 i_4}(1,2,3,4;|\Psi\rangle) - M_{i_1 i_2}(1,2;|\Psi\rangle)M_{i_3 i_4}(3,4;|\Psi\rangle) \\ -M_{i_1 i_3}(1,3;|\Psi\rangle)M_{i_2 i_4}(2,4;|\Psi\rangle) - M_{i_1 i_4}(1,4;|\Psi\rangle)M_{i_2 i_3}(2,3;|\Psi\rangle). \quad (5)$$

This provides the complete criterion for multipartite entanglement which includes the multi-separable subsystems. If $M'_{i_1 i_2 \cdots i_n}(\alpha_1, \alpha_2, \cdots, \alpha_N; |\Psi\rangle) = 0$ for any $i_1, i_2, \cdots, i_n$, the given state is separable, and has two possibilities. The first is that every term on the right hand side in eq. (5) is zero. This means that the state has the from of $|\Psi(1,2,3,4)\rangle = |\psi(\alpha_1, \alpha_2, \alpha_3)\rangle \otimes |\phi(\alpha_4)\rangle$. The second is that the first term, $M_{i_1 i_2 i_3 i_4}(1,2,3,4;|\Psi\rangle)$, is subtracted by three terms with a negative sign. However, we can show without difficulties that only one term out of three terms with the negative sign is nonzero; if $M_{ij}(1,2,;|\Psi\rangle)M_{kl}(3,4;|\Psi\rangle) \neq 0$ for the four-qubit case, $M_{i_1 i_2 i_3 i_4}(1,2,3,4;|\Psi\rangle) = M_{ij}(1,2;|\Psi\rangle)M_{kl}(3,4;|\Psi\rangle)$. This fact lead us to find that the state is $|\Psi(1,2,3,4)\rangle = |\psi(1,2)\rangle \otimes |\phi(3,4)\rangle$.

If any pure state of a N-qubit system is given, we must, at first, check whether $M'_{i_1 i_2 \cdots i_N}(\alpha_1, \alpha_2, \cdots, \alpha_N; |\Psi\rangle)$ is zero or not. In the nonzero case, the N-qubit is totally entangled but in the zero case, the state is either completely separated or partially entangled, as described earlier. In the zero case, we have to make additional checks whether $M'_{i_1, i_2, \cdots 1_n}(\alpha_1, \alpha_2, \cdots, \alpha_n; |\Psi\rangle)$ for $n < N$ is zero or not in sequence. These sequential checks determine whether the given state is totally entangled, biseparable, triseparable, $\cdots$ or completely separable. Then M' classifies all the possible forms of entangled states.

By the tensor, M', we can classify the pure multiqubit states including many different entanglement types. However, the tensor cannot distinguish the entangled states which are connected to each others under a local unitary transformation. For instance, M' can distinguish the product state from the Bell states in a two-qubit system. However, the values of M' of four Bell states are different from each others. One may misunderstand that the four Bell states have different degrees of entanglements. It is well known that the four Bell states are equivalent under a local unitary transformation as maximally entangled states. M' just distinguishes whether the multiqubit state are entangled or multiseparated. Therefore, we need a new quantity to determine an entanglement magnitude.

Based on M', we introduce a measure of an entanglement such as

$$B^{(m)}(\alpha_1,\alpha_2,\cdots\alpha_m;|\Psi\rangle)=\sum_{ijkl\cdots}M'_{ijkl\cdots}(\alpha_1,\alpha_2,\cdots\alpha_m;|\Psi\rangle)M'_{ijkl\cdots}(\alpha_1,\alpha_2,\cdots\alpha_m;|\Psi\rangle). \tag{6}$$

$B^{(m)}(\alpha_1,\alpha_2,\cdots\alpha_m;|\Psi\rangle)$ calculates the entanglement magnitude among m qubits labeled by $\alpha_1,\alpha_2,\cdots\alpha_m$. For example, $B^{(2)}(\alpha_1,\alpha_2;|\Psi\rangle)$ describes the entanglement magnitude between the qubits α_1 and α_2 and $B^{(3)}(\alpha_1,\alpha_2,\alpha_3;|\Psi\rangle)$ the entanglement degree among the qubits $\alpha_1,\alpha_2,\alpha_3$. $B^{(2)}(1,2;|\Psi\rangle)$ in two-qubit systems is the same measure as Schlinz and Mahler's entanglement measure[14].

$B^{(m)}$ of eq. (6) satisfies the following properties affirming entanglement monotone, First, it is nonnegative, because B is defined by the square of real numbers and $M'=0$ in separable states. Second, it is invariant under any local unitary transformations. This can be shown easily by using $U^\dagger\sigma_i U=T_{ij}\sigma_j$ and $\sum_i T_{ij}T_{ik}=\delta_{jk}$ where U is a unitary matrix and T is a 3×3 orthogonal matrix in the qubit systems[14]. Third, it is nonincreasing under local measurements. The measurement collapses an entangled n-qubit state to biseparable state that one qubit under local measurement processes are uncorrelated with the others. Then, $B^{(m)}$ is vanished under local measurements and less than that of the original state. Fourth, it is invariant under the addition of an uncorrelated ancillary state. Fifth, it is not increased by tracing out a part of the system. Thus we claim that $B^{(m)}$ is the entanglement monotone.

For example, we consider four-qubit states which are totally entangled,

$$\begin{aligned}
|GHZ_4\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle+|1111\rangle,\\
|W_4\rangle &= \frac{1}{2}(|0001\rangle+|0010\rangle+|0100\rangle+|1000\rangle),\\
|\phi_6\rangle &= \frac{1}{\sqrt{6}}(|0011\rangle+|0101\rangle+|1001\rangle+|1010\rangle+|0110\rangle+|1100\rangle)\\
|\phi_4\rangle &= \frac{1}{2}(|0000\rangle+|0011\rangle+|1100\rangle-|1111\rangle).
\end{aligned} \tag{7}$$

We can get $M'_{ijkl}(1,2,3,4;|\Psi\rangle)\neq0$ for the above four states and know that their states are totally entangled. Then we can distinguish the entanglement difference through calculation of $B^{(4)}$; $B^{(4)}(1,2,3,4;|GHZ_4\rangle)=1$, $B^{(4)}(1,2,3,4;|W_4\rangle)=\frac{51}{256}$, $B^{(4)}(1,2,3,4;|\phi_6\rangle)=\frac{7}{27}$, and $B^{(4)}(1,2,3,4;|\phi_4\rangle)=\frac{1}{3}$. Here we normalized $B^{(4)}$ with the value of the GHZ state. A partially

three-qubit entangled state as $|GHZ_3\rangle \otimes |0\rangle$ and a partially two-qubit entangled state as $|Bell_2\rangle \otimes |Bell_2\rangle$ where $|Bell_2\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$ have $M' = 0$ for all subindexes. Then this explains these states do not have total entanglement. However, we can obtain $M'(1,2,3)$ is not zero for $|GHZ_3\rangle \otimes |0\rangle$ and $M'(1,2)$ and $M'(3,4)$ are not zero for $|Bell_2\rangle \otimes |Bell_2\rangle$. This shows that $|GHZ_3\rangle \otimes |0\rangle$ has a partial entanglement among qubits $1,2$ and 3 and $|Bell_2\rangle \otimes |Bell_2\rangle$ has partial entanglements between qubits 1 and 2, and between qubits 3 and 4.

If the Pauli's operators of the eq. (2) which are the generators of $SU(2)$ unitary group are replaced by the generators related to the higher dimensional Hilbert space, the same method can be also applied to the higher dimensional composite systems.

Here we have mainly focused on the pure systems. In the mixed case, it is not easy to apply directly as

$$\begin{aligned} M_{i_1 i_2 \cdots i_n}(1,2,\cdots,n;\rho) &= tr[\rho(\sigma_{i_1}(1) - \lambda_i(1)) \otimes (\sigma_{i_2}(2) - \lambda_{i_2}(2)) \otimes \cdots \\ &\quad \otimes(\sigma_{i_n}(n) - \lambda_{i_n}(n))], \end{aligned} \tag{8}$$

where ρ is the density operator. In the Werner state of two-qubit parameterized by fidelity with the singlet states, we get $M' \neq 0$ for the region $F \leq \frac{1}{2}$. This contradicts the fact that the Werner state is separable in the region $F \leq \frac{1}{2}$[10]. It is why the density operator has many possible ensembles for the given density operator. We have to find the optimized ensemble to apply the criterion. If there exists a pure state ensemble to make M' zero, the density operator is separable. Otherwise, the density operator is entangled. However, this story seems to be simple in principle but it is very complicated to find the proper ensemble practically.

We have here presented the classification and quantification scheme of a general multipartite systems. M' which is defined by the expectation values of $SU(N)$ generators determines the entangled type of multipartite states without a priori knowledge on the quantum states investigated. However, the same entangled type has many different states which cannot distinguish with M' any more. So we have introduced $B^{(m)}$ providing the quantification procedure. Finally, we have shown how to apply our method to the four-qubit system as an example.

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References

[1] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881(1992).

[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wooters, Phys. Rev. Lett. **70**, 1895(1993).

[3] I. L. Chuang, Phys. Rev. Lett. **85**, 2006(2000).

[4] R. Jozsa, D. S. Abrams, J. D. Dowling and C. P. Williams, Phys. Rev. Lett. **85**, 2010(2000).

[5] A. K. Ekert, Phys. Rev. Lett. **67**, 661(1991).

[6] A. Peres, Phys. Rev. Lett. **77**,1413(1996).

[7] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. **A223**, 1(1996).

[8] W. K. Wooters, Quant. Inf. Comp. **1**, 27(2001).

[9] G. Vidal and R. F. Werner, Phys. Rev. **A65**, 032314(2002).

[10] C. H. Bennett, D. P. DiVincenzo, J. Smolin and W. K. Wootters, Phys. Rev. **A54**, 3824(1996).

[11] H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 5188(2001)

[12] V. Coffman, J. Kundu and W. K. Wootters, Phys.Rev. **A61**, 052306(2000).

[13] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Guehne, P. Hyllus, D. Bruss, M. Lewenstein, A. Sanpera, Phys. Rev. Lett. **92**, 087902 (2004)

[14] J. Schlinz and G. Mahler, Phys. Rev. **A52**, 4396(1995).